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Semisimple Corings

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Abstract. While semisimple artinian rings and semisimple coalgebras over a field are described in terms of matrices, semisimple corings seem to have a more intricate structure. Some properties of semisimple rings or coalgebras, which are deduced from the aforementioned structure, are not evident over a (left) semisimple coring. For instance, it is not evident whether the notion of semisimple corings is left-right symmetric. We develop the basic essentials for a theory of semisimple corings, giving a positive answer for the last question, as well as some information about the structure of semisimple corings.

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Sweedler [14] introduced the notion of corings as a generalization of the concept of coalgebras in order to study the set of intermediate division rings for an extension of division rings. It turns out that this formalism embodies several kinds of relative module categories. Thus, graded modules, Doi–Hopf modules, and more generally, entwined modules are instances of comodules over suitable corings (see [4] and its references). From this point of view, an interesting question is to characterize those corings encoding the simplest type of category of comodules. Since, under certain hypotheses, the category of comodules is abelian, the simple objects play a relevant role in its structure. In the most favorable case, all comodules are direct sums of simple comodules (i.e., the category of comodules is semisimple). In the classical theory of modules over rings, the study of semisimple rings precedes the development of the entire theory. This paper states the basic essentials for a theory of semisimple corings.

Throughout this paper, the word ring will refer to an associative algebra over a commutative ring K, and the term subring is then understood as a subalgebra. The category of all left (resp., right) modules over a ring R will be denoted by ${}_R\mathcal{M}$ (resp., $\mathcal{M}_R$). An agile introduction to abelian categories is contained in [13]. The notation $X \in \mathcal{A}$ for a category $\mathcal{A}$ means that X is an object of $\mathcal{A}$, and the identity morphism attached to any object X will be represented by the object itself.

1 Corings and Comodules

We discuss under which conditions the category of (right) comodules over a coring is abelian. We first recall from [14] the notion of corings.

Let A denote a ring. An *A-coring* is a triple $(\mathfrak{C}, \Delta, \epsilon)$ consisting of an A-bimodule $\mathfrak{C}$ and two A-bimodule maps

$$\Delta : \mathfrak{C} \longrightarrow \mathfrak{C} \otimes_A \mathfrak{C}, \qquad \epsilon : \mathfrak{C} \longrightarrow A$$

such that the diagrams

$$\begin{array}{ccc} \mathfrak{C} & \xrightarrow{\Delta} & \mathfrak{C} \otimes_A \mathfrak{C} \\ {\scriptstyle\Delta}\downarrow & & \downarrow{\scriptstyle \mathfrak{C} \otimes_A \Delta} \\ \mathfrak{C} \otimes_A \mathfrak{C} & \xrightarrow{\Delta \otimes_A \mathfrak{C}} & \mathfrak{C} \otimes_A \mathfrak{C} \otimes_A \mathfrak{C} \end{array}$$

and

$$\begin{array}{ccc} \mathfrak{C} & \xrightarrow{\Delta} & \mathfrak{C} \otimes_A \mathfrak{C} \\ & {\scriptstyle\cong}\searrow & \downarrow{\scriptstyle \mathfrak{C} \otimes_A \epsilon} \\ & & \mathfrak{C} \otimes_A A \end{array} \qquad \begin{array}{ccc} \mathfrak{C} & \xrightarrow{\Delta} & \mathfrak{C} \otimes_A \mathfrak{C} \\ & {\scriptstyle\cong}\searrow & \downarrow{\scriptstyle \epsilon \otimes_A \mathfrak{C}} \\ & & A \otimes_A \mathfrak{C} \end{array}$$

commute.

A *left* $\mathfrak{C}$*-comodule* is a pair (M, λ_M) consisting of a left A-module M and an A-linear map $\lambda_M : M \to \mathfrak{C} \otimes_A M$ such that the diagrams

$$\begin{array}{ccc} M & \xrightarrow{\lambda_M} & \mathfrak{C} \otimes_A M \\ {\scriptstyle\lambda_M}\downarrow & & \downarrow{\scriptstyle \Delta \otimes_A M} \\ \mathfrak{C} \otimes_A M & \xrightarrow{\mathfrak{C} \otimes_A \lambda_M} & \mathfrak{C} \otimes_A \mathfrak{C} \otimes_A M \end{array} \qquad \begin{array}{ccc} M & \xrightarrow{\lambda_M} & \mathfrak{C} \otimes_A M \\ & {\scriptstyle\cong}\searrow & \downarrow{\scriptstyle \epsilon \otimes_A M} \\ & & A \otimes_A M \end{array}$$

commute. Right $\mathfrak{C}$-comodules are similarly defined; we use the notation ρ_M for their structure maps. A *morphism* of left $\mathfrak{C}$-comodules (M, λ_M) and (N, λ_N) is an A-linear map $f : M \to N$ such that the following diagram is commutative:

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ {\scriptstyle\lambda_M}\downarrow & & \downarrow{\scriptstyle\lambda_N} \\ \mathfrak{C} \otimes_A M & \xrightarrow{\mathfrak{C} \otimes_A f} & \mathfrak{C} \otimes_A N \end{array}$$

The K-module of all left $\mathfrak{C}$-comodule morphisms from M to N is denoted by $\mathrm{Hom}_{\mathfrak{C}}(M, N)$. The category of all left $\mathfrak{C}$-comodules will be denoted by ${}^{\mathfrak{C}}\mathcal{M}$. Analogously, we can consider the category of all right $\mathfrak{C}$-comodules $\mathcal{M}^{\mathfrak{C}}$. Every valid statement about left comodules entails a correct assertion for right comodules, which will be implicitly understood.

Coproducts and cokernels in ${}^{\mathfrak{C}}\mathcal{M}$ do exist, and they can be already computed in ${}_A\mathcal{M}$. Therefore, ${}^{\mathfrak{C}}\mathcal{M}$ has arbitrary inductive limits. If $\mathfrak{C}_A$ is a flat module, then ${}^{\mathfrak{C}}\mathcal{M}$ is easily proved to be an abelian category. The converse is not true as the following example shows.

Example 1.1. Let $A = \begin{pmatrix} R & B \\ 0 & S \end{pmatrix}$ be a generalized triangular matrix ring with B an (R, S)-bimodule over the rings R and S. It is well known that the right A-modules are given by triples $M = (M', M'', \mu)$ consisting of a right R-module M', a right S-module M'' and an S-module map $\mu : M' \otimes_R B \to M''$. A homomorphism of right A-modules is then given by a pair $f = (f', f'') : (M', M'', \mu) \to (N', N'', \nu)$ consisting of an R-module map $f' : M' \to N'$ and a right S-module map $f'' : M'' \to N''$ such that $f''\mu = \nu(f' \otimes_R B)$. Consider the ideal $I = \begin{pmatrix} R & B \\ 0 & 0 \end{pmatrix}$ of A, which corresponds to (R, B, μ) as a right A-module, where $\mu : R \otimes_R B \to B$ is the canonical isomorphism. Now $M \otimes_A I = (M', M' \otimes_R B, id)$ and the multiplication map $M \otimes_A I \to M$ is given by $(id, \mu) : (M', M' \otimes_R B, id) \to (M', M'', \mu)$. The A-bimodule I is an A-coring with comultiplication given by the isomorphism $I \cong I \otimes_A I$ and counit given by the inclusion $I \subseteq A$. It can be easily shown that a right A-linear coaction $\rho_M = (\rho', \rho'') : M \to M \otimes_A I$ is an I-comodule structure if and only if $\rho' = id_{M'}$ and μ is an isomorphism with inverse ρ''. Therefore, the category $\mathcal{M}^I$ of all right I-comodules can be identified with the category of all right A-modules (M', M'', μ) such that μ is an isomorphism. The functor $F : \mathcal{M}^I \to \mathcal{M}_R$ given by $F(M', M'', \mu) = M'$ is easily shown to be an equivalence of categories. In particular, $\mathcal{M}^I$ is a Grothendieck category and ${}_AI$ is not flat unless ${}_RB$ is.

The following result clarifies the situation created by our example. Recall from [9, Proposition 3.1] that the functor $\mathfrak{C} \otimes_A - : {}_A\mathcal{M} \to {}^{\mathfrak{C}}\mathcal{M}$ is right adjoint to the forgetful functor $U : {}^{\mathfrak{C}}\mathcal{M} \to {}_A\mathcal{M}$.

Proposition 1.2. *Let $\mathfrak{C}$ be an A-coring. Then the following are equivalent:*

(i) *$\mathfrak{C}_A$ is flat.*

(ii) *${}^{\mathfrak{C}}\mathcal{M}$ is an abelian category and the functor U is left exact.*

(iii) *${}^{\mathfrak{C}}\mathcal{M}$ is a Grothendieck category and the functor U is left exact.*

Proof. (i)⇒(iii). The exactness of the functor $\mathfrak{C} \otimes_A - : {}_A\mathcal{M} \to {}_A\mathcal{M}$ entails that kernels in ${}^{\mathfrak{C}}\mathcal{M}$ can be already computed in ${}_A\mathcal{M}$. This gives that ${}^{\mathfrak{C}}\mathcal{M}$ is a complete and co-complete abelian category with exact direct limits. We need to find a generator for ${}^{\mathfrak{C}}\mathcal{M}$. For this, we proceed as in the proof of

[15, 13.13]. Let $M \in {}^{\mathfrak{C}}\mathcal{M}$ with coaction λ_M, and $A^{(I)} \to M \to 0$ the free presentation of M in ${}_A\mathcal{M}$. We have

$$\begin{array}{ccccc}
\mathfrak{C}^{(I)} \cong \mathfrak{C} \otimes_A A^{(I)} & \xrightarrow{\ g\ } & \mathfrak{C} \otimes_A M & \longrightarrow & 0 \\
\uparrow & & \uparrow{\scriptstyle \lambda_M} & & \\
g^{-1}(M) & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

Now it is clear that

$$\mathfrak{g}^l = \oplus \{ L \mid L \text{ is a left subcomodule of } \mathfrak{C}^k,\ k \in \mathbb{N} \}$$

is a generator of ${}^{\mathfrak{C}}\mathcal{M}$. Obviously, the forgetful functor $U : {}^{\mathfrak{C}}\mathcal{M} \to {}_A\mathcal{M}$ is exact in this case.

(iii)$\Rightarrow$(ii). This is obvious.

(ii)$\Rightarrow$(i). By [12, Corollary 3.2.3], $\mathfrak{C} \otimes_A - : {}_A\mathcal{M} \to {}^{\mathfrak{C}}\mathcal{M}$ is left exact, and thus, $U \circ (\mathfrak{C} \otimes_A -) : {}_A\mathcal{M} \to {}_A\mathcal{M}$ is left exact too. Therefore, $\mathfrak{C}_A$ is a flat module. □

As a consequence of the proof of Proposition 1.2, we obtain:

Corollary 1.3. *If $\mathfrak{C}_A$ is flat, then every left $\mathfrak{C}$-comodule is isomorphic to a subcomodule of a $\mathfrak{C}$-generated left comodule.*

Example 1.4. [14, Example 1.2] Let $B \to A$ be a ring extension. Then $\mathfrak{C} = A \otimes_B A$ is an A-coring with coproduct $\mathfrak{C} \to \mathfrak{C} \otimes_A \mathfrak{C}$ given by $a \otimes_B a' \mapsto a \otimes_B 1 \otimes_A 1 \otimes_B a'$ and counit $A \otimes_B A \to A$ the multiplication map.

Example 1.5. [4, Proposition 2.2] Let (C, Δ, ϵ) be a K-coalgebra and assume the canonical left A-module $\mathfrak{C} = A \otimes C$ has a right A-module structure which makes $\mathfrak{C}$ an A-bimodule. Define the K-linear map $\psi : C \otimes A \to A \otimes C$ as $\psi(c \otimes a) = (1 \otimes c)a$. Consider the left A-module maps $\Delta_{\mathfrak{C}} : \mathfrak{C} \to \mathfrak{C} \otimes_A \mathfrak{C} \cong A \otimes C \otimes C$, $\Delta_{\mathfrak{C}} = A \otimes \Delta$, and $\epsilon_{\mathfrak{C}} = A \otimes \epsilon$. Then $(\mathfrak{C}, \Delta_{\mathfrak{C}}, \epsilon_{\mathfrak{C}})$ is an A-coring if and only if $(A, C)_\psi$ is an entwining structure (see [5]). Moreover, $\mathcal{M}^{\mathfrak{C}}$ is isomorphic to the category $\mathcal{M}_A^C(\psi)$ of entwined modules. Examples of categories of entwined modules are Doi–Koppinen modules, introduced in [6] and [10] (cf. [3, Example 3.1(3)]).

2 Rational Modules and Comodules

We state a formal framework (the notion of rational pairing) which reduces the study of some categories of comodules to the investigation of certain subcategories of categories of modules. The development is adapted from that given in [8] and [2] for coalgebras over commutative rings, and we include only those proofs which do not follow closely to that given in [8] for similar results. Some of these results have been also obtained in [1] and [16]. We also include a discussion on the relationship between bicomodules and bimodules which is essential in our approach to the study of semisimple and simple corings. The main results in this direction (Proposition 2.10 and Corollary 2.11) are new even for coalgebras over commutative rings.

Let P and Q be A-bimodules. Any balanced bilinear form

$$\langle -,- \rangle : P \times Q \longrightarrow A$$

provides natural transformations $\alpha : Q \otimes_A - \rightarrow \mathrm{Hom}_A({}_AP, -)$ and $\beta : - \otimes_A P \rightarrow \mathrm{Hom}_A(Q_A, -)$ given by

$$\begin{aligned} \alpha_M : Q \otimes_A M &\longrightarrow \mathrm{Hom}_A({}_AP, {}_AM), \\ q \otimes_A m &\longmapsto [p \mapsto \langle p, q \rangle m], \\ \beta_N : N \otimes_A P &\longrightarrow \mathrm{Hom}_A(Q_A, N_A), \\ n \otimes_A p &\longmapsto [q \mapsto n \langle p, q \rangle]. \end{aligned}$$

Moreover, if M is an A-bimodule, then α_M and β_M are bimodule homomorphisms. The canonical isomorphisms provide two bimodule maps

$$\begin{aligned} \alpha_A : Q &\longrightarrow \mathrm{Hom}_A({}_AP, {}_AA) = {}^*P, \\ q &\longmapsto [p \mapsto \langle p, q \rangle], \end{aligned} \tag{1}$$

$$\begin{aligned} \beta_A : P &\longrightarrow \mathrm{Hom}_A(Q_A, A_A) = Q^*, \\ p &\longmapsto [q \mapsto \langle p, q \rangle], \end{aligned} \tag{2}$$

which are bimodule homomorphisms. So we can recover the balanced bilinear form if one of the natural transformations is given.

Definition 2.1. The data $T = (P, Q, \langle -,- \rangle)$ are called a *left rational system* if α_M is injective for each left A-module M, and a *right rational system* if β_N is injective for every right A-module N.

Remark 2.2. [2, Remark 2.4] Let $(P, Q, \langle -,- \rangle)$ be a left rational system. Let $M \in {}_A\mathcal{M}$, and N a submodule of M with the canonical injection i_N. Consider the following commutative diagram:

$$\begin{array}{ccc} Q \otimes_A N & \xrightarrow{\alpha_N} & \mathrm{Hom}_A(P, N) \\ {\scriptstyle Q \otimes_A i_N} \downarrow & & \downarrow {\scriptstyle i} \\ Q \otimes_A M & \xrightarrow{\alpha_M} & \mathrm{Hom}_A(P, M) \end{array}$$

Hence, $Q \otimes_A i_N$ is injective. Since M was an arbitrary left A-module, we conclude that Q_A should be a flat A-module. Analogously, if $(P, Q, \langle -,- \rangle)$ is a right rational system, then ${}_AP$ is flat.

Let $\langle -,- \rangle : P \times Q \rightarrow A$ and $[-,-] : P' \times Q' \rightarrow A$ be two balanced bilinear forms with natural transformations α, β and α', β', respectively. We can define a new balanced bilinear form

$$\{-,-\} : (P \otimes_A P') \times (Q' \otimes_A Q) \longrightarrow A$$

given by

$$(p \otimes_A p', q' \otimes_A q) \longmapsto \{p \otimes_A p', q' \otimes_A q\} = \langle p, [p', q']q \rangle = \langle p[p', q'], q \rangle.$$

The natural transformations associated to $\{-,-\}$ are given by the compositions

$$\begin{array}{ccc} Q' \otimes_A Q \otimes_A M & \longrightarrow & \mathrm{Hom}_A(P \otimes_A P',\, M) \\ {\scriptstyle \alpha'_{Q \otimes_A M}} \downarrow & & \uparrow {\scriptstyle i_M} \\ \mathrm{Hom}_A({}_AP',\, Q \otimes_A M) & \xrightarrow{(\alpha_M)_*} & \mathrm{Hom}_A({}_AP',\, \mathrm{Hom}_A({}_AP, M)) \end{array}$$

$$\begin{array}{ccc} N \otimes_A P \otimes_A P' & \longrightarrow & \mathrm{Hom}_A(Q' \otimes_A Q,\, N) \\ {\scriptstyle \beta'_{N \otimes_A P}} \downarrow & & \uparrow {\scriptstyle i_N} \\ \mathrm{Hom}_A(Q'_A,\, N \otimes_A P) & \xrightarrow{(\beta_N)_*} & \mathrm{Hom}_A(Q'_A,\, \mathrm{Hom}_A(Q_A, M)) \end{array}$$

The following proposition, which is now clear, replaces [8, Proposition 2.2] in order to show that the canonical comodule structure over a rational module is pseudo-coassociative.

Proposition 2.3. *Let $(P, Q, \langle -,-\rangle)$ and $(P', Q', [-,-])$ be two left (resp., right) rational systems. Then $(P \otimes_A P', Q' \otimes_A Q, \{-,-\})$ is also a left (resp., right) rational system.*

Let $(\mathfrak{C}, \Delta, \epsilon)$ be an A-coring. Recall that $\mathfrak{C}^* = \mathrm{Hom}_A(\mathfrak{C}_A, A)$ (resp., ${}^*\mathfrak{C} = \mathrm{Hom}_A({}_A\mathfrak{C}, A)$) is a ring extension of A^{op} with multiplication $gf = f \circ (g \otimes_A \mathfrak{C}) \circ \Delta$ (resp., $gf = g \circ (\mathfrak{C} \otimes_A f) \circ \Delta$). Both units are ϵ. See [14, Proposition 3.2] for details.

Definition 2.4. A *left rational pairing* is a left rational system $(B, \mathfrak{C}, \langle -,-\rangle)$ such that B is a ring extension of A, $\mathfrak{C}$ is an A-coring, and $\beta : B \to \mathfrak{C}^*$ is a ring antimorphism. If $\Delta(c) = \sum_i c_i \otimes_A d_i$, then

$$\langle ab, c \rangle = \left\langle a, \sum_i \langle b, c_i \rangle d_i \right\rangle \;\; \forall a, b \in B \quad \text{and} \quad \epsilon = \langle 1, - \rangle. \tag{3}$$

Analogously, a *right rational pairing* is a right rational system $(\mathfrak{C}, B', [-,-])$ such that B' is a ring extension of A, $\mathfrak{C}$ is an A-coring, and $\alpha : B' \to {}^*\mathfrak{C}$ is a ring antimorphism. If $\Delta(c) = \sum_i c_i \otimes_A d_i$, then

$$[c, ab] = \left[\sum_i c_i [d_i, a],\, b \right] \;\; \forall a, b \in B \quad \text{and} \quad \epsilon = [1, -].$$

Example 2.5. Let $\mathfrak{C}$ be an A-coring such that $\mathfrak{C}$ is projective as a right A-module. By using any dual basis associated with the projectivity of $\mathfrak{C}_A$, we prove that the canonical balanced bilinear form $\langle -,- \rangle : \mathfrak{C}^{*op} \times \mathfrak{C} \to A$ gives a left rational pairing $T = (\mathfrak{C}^{*op}, \mathfrak{C}, \langle -,- \rangle)$. Analogously, if $\mathfrak{C}$ is an A-coring such that ${}_A\mathfrak{C}$ is a projective module, then $T' = (\mathfrak{C}, {}^*\mathfrak{C}^{op}, [-,-])$ is a right rational pairing.

Let $T = (B, \mathfrak{C}, \langle -,- \rangle)$ be a left rational pairing. An element m in a left B-module M is called *rational* if there exists a set of *left rational parameters* $\{(c_i, m_i)\} \subseteq \mathfrak{C} \times M$ such that $bm = \sum_i \langle b, c_i \rangle m_i$ for all $b \in B$. The set of rational elements in M is denoted by $\mathrm{Rat}^T(M)$. The proofs detailed in [8, Section 2] can be adapted in a straightforward way in order to show that $\mathrm{Rat}^T(M)$ is a B-submodule of M and the assignment $M \mapsto \mathrm{Rat}^T(M)$ defines a functor

$$\mathrm{Rat}^T : {}_B\mathcal{M} \to {}_B\mathcal{M},$$

which is in fact a left exact preradical. Therefore, the full subcategory $\mathrm{Rat}^T({}_B\mathcal{M})$ of ${}_B\mathcal{M}$ whose objects are those B-modules M such that $\mathrm{Rat}^T(M) = M$ is a closed reflective subcategory [7, p. 395], and in particular, it is a Grothendieck category. The modules in the subcategory $\mathrm{Rat}^T({}_B\mathcal{M})$ will be called *rational left B-modules* (with respect to T). Now it turns out that every rational left B-module is a left $\mathfrak{C}$-comodule with structure map $\lambda_M : M \to \mathfrak{C} \otimes_A M$ given by $\lambda_M(m) = \sum c_i \otimes_A m_i$, where $\{(c_i, m_i)\}$ is any set of rational parameters for $m \in M$ (see [8, Proposition 3.5] for a proof which can be adapted to the present setting). This leads to a functor

$$^{\mathfrak{C}}(-) : \mathrm{Rat}^T({}_B\mathcal{M}) \longrightarrow {}^{\mathfrak{C}}\mathcal{M},$$

which can be shown to be an isomorphism of categories with the guide of [8, Section 3]. It can be also deduced that ${}_B\mathfrak{C}$ becomes a subgenerator for $\mathrm{Rat}^T({}_B\mathcal{M})$. Therefore, we can state

Theorem 2.6. *Let $T = (B, \mathfrak{C}, \langle -,- \rangle)$ be a left rational pairing. The functor $^{\mathfrak{C}}(-) : \mathrm{Rat}^T({}_B\mathcal{M}) \to {}^{\mathfrak{C}}\mathcal{M}$ is an isomorphism of categories. Moreover, every left $\mathfrak{C}$-comodule is isomorphic to a B-submodule of a ${}_B\mathfrak{C}$-generated B-module.*

This theorem when applied to the rational pairing $T = (\mathfrak{C}^{*op}, \mathfrak{C}, \langle -,- \rangle)$ given in Example 2.5 leads to

Corollary 2.7. *Let $\mathfrak{C}$ be an A-coring. If $\mathfrak{C}$ is projective as a right A-module, then the functor $(-)^{\mathfrak{C}} : \mathrm{Rat}^T(\mathcal{M}_{\mathfrak{C}^*}) \to {}^{\mathfrak{C}}\mathcal{M}$ is an isomorphism of categories. Moreover, every left $\mathfrak{C}$-comodule is isomorphic to a $\mathfrak{C}^*$-submodule of a $\mathfrak{C}_{\mathfrak{C}^*}$-generated $\mathfrak{C}^*$-module.*

Theorem 2.6 has a right analogue. If $T' = (\mathfrak{C}, B', [-,-])$ is a right rational pairing, then we can define functors $\mathrm{Rat}^{T'} : \mathcal{M}_{B'} \to \mathcal{M}_{B'}$ and $(-)^{\mathfrak{C}} : \mathrm{Rat}^{T'}(\mathcal{M}_{B'}) \to \mathcal{M}^{\mathfrak{C}}$. These functors lead to the following theorem.

Theorem 2.6′. *Let $T' = (\mathfrak{C}, B', [-,-])$ be a right rational pairing. The functor $(-)^{\mathfrak{C}} : \mathrm{Rat}^{T'}(\mathcal{M}_{B'}) \to \mathcal{M}^{\mathfrak{C}}$ is an isomorphism of categories. Moreover, every right $\mathfrak{C}$-comodule is isomorphic to a B'-submodule of a $\mathfrak{C}_{B'}$-generated B'-module.*

Finally, we state a useful consequence of the former development.

Proposition 2.8. *Let $T = (B, \mathfrak{C}, \langle -,- \rangle)$ (resp., $T' = (\mathfrak{C}, B', [-,-])$) be a left (resp., right) rational pairing. Let $M \in {}^{\mathfrak{C}}\mathcal{M}$. Then M is a finitely generated left (resp., right) $\mathfrak{C}$-comodule if and only if M is a finitely generated left (resp., right) A-module.*

Recall that a $\mathfrak{C}$-bicomodule is an A-bimodule M endowed with a right A-linear left $\mathfrak{C}$-comodule structure $\lambda_M : M \to \mathfrak{C} \otimes_A M$ and a left A-linear right $\mathfrak{C}$-comodule structure $\rho_M : M \to M \otimes_A \mathfrak{C}$ such that

$$(\lambda_M \otimes_A \mathfrak{C})\rho_M = (\mathfrak{C} \otimes_A \rho_M)\lambda_M. \tag{4}$$

The $\mathfrak{C}$-bicomodules are the objects of a category ${}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{C}}$ whose morphisms are those A-bimodule homomorphisms which are left and right $\mathfrak{C}$-colinear.

Let $T = (B, \mathfrak{C}, \langle -,- \rangle)$ (resp., $T' = (\mathfrak{C}, B', [-,-])$) be a left (resp., right) rational pairing.

Lemma 2.9. *Let M be an A-bimodule with a left $\mathfrak{C}$-comodule structure $\lambda_M : M \to \mathfrak{C} \otimes_A M$ and a right $\mathfrak{C}$-structure map $\rho_M : M \to M \otimes_A \mathfrak{C}$. Then M is a $\mathfrak{C}$-bicomodule if and only if M is a (B, B')-bimodule.*

Proof. By Theorems 2.6 and 2.6′, M is a rational left B-module and a rational right B'-module. We first prove that λ_M is right A-linear if and only if M is a (B, A)-bimodule. For each $m \in M$, write $\lambda_M(m) = \sum c_i \otimes_A m_i$ for a set of left rational parameters $\{(c_i, m_i)\}$. Thus, λ_M is right A-linear if and only if $\{(c_i, m_i a)\}$ is a set of rational parameters for ma for our generic m and every $a \in A$. But this last condition is easily proved to be equivalent to that M is a (B, A)-bimodule. Of course, ρ_M is left A-linear if and only if M is an (A, B')-bimodule. Thus, we see that in order to prove the lemma, we can assume M is a (B, A)-bimodule and an (A, B')-bimodule. Under this condition, M is a $\mathfrak{C}$-bicomodule if and only if

$$\sum c_i \otimes_A m_{ij} \otimes_A d_{ij} = \sum c_{ji} \otimes_A m_{ji} \otimes_A d_j,$$

where $\{(c_i, m_i)\}$ is a set of left rational parameters for m, $\{(m_{ij}, d_{ij})\}$ is a set of right rational parameters for each m_i, $\{(m_j, d_j)\}$ is a set of right rational parameters for m, and $\{(c_{ji}, m_{ji})\}$ is a set of left rational parameters for each m_j. An easy computation gives $(b.m).b' = \sum \langle b, c_i \rangle m_{ij} [d_{ij}, b']$ and $b.(m.b') = \sum \langle b, c_{ji} \rangle m_{ji} [d_j, b']$ for any $(b, b') \in B \times B'$. Hence, M is a (B, B')-bimodule if and only if

$$\sum \langle b, c_i \rangle m_{ij} [d_{ij}, b'] = \sum \langle b, c_{ji} \rangle m_{ji} [d_j, b'] \quad \forall (b, b') \in B \times B'.$$

By Remark 2.2, the following map is injective

$$\mathfrak{C}\otimes_A M\otimes_A \mathfrak{C} \xhookrightarrow{\mathfrak{C}\otimes_A \beta_M} \mathfrak{C}\otimes_A \mathrm{Hom}_A(B'_A, M) \xhookrightarrow{\alpha_{\mathrm{Hom}_A(B'_A,M)}} \mathrm{Hom}_A({}_AB, \mathrm{Hom}_A(B'_A, M)).$$

Hence, M is a $\mathfrak{C}$-bicomodule if and only if it is a (B, B')-bimodule. □

Proposition 2.10. *Let $T = (B, \mathfrak{C}, \langle -,-\rangle)$ and $T' = (\mathfrak{C}, B', [-,-])$ be rational pairings, and $\mathrm{Rat}^{T,T'}({}_B\mathcal{M}_{B'})$ the full subcategory of the category ${}_B\mathcal{M}_{B'}$ whose objects are the (B, B')-bimodules which are rational as B-modules and as B'-modules. Then there is an isomorphism of categories $\mathrm{Rat}^{T,T'}({}_B\mathcal{M}_{B'}) \cong {}^{\mathfrak{C}}\mathcal{M}^{\mathfrak{C}}$.*

Proof. If M is a $\mathfrak{C}$-bicomodule, then by Lemma 2.9, M is a (B, B')-bimodule, and by Theorems 2.6 and 2.6′, M is rational as a left B-module and as a right B-module. Conversely, every (B, B')-bimodule M such that the modules ${}_BM$ and $M_{B'}$ are rational is a $\mathfrak{C}$-bicomodule by Lemma 2.9 and Theorems 2.6 and 2.6′. □

Corollary 2.11. *Let $\mathfrak{I}$ be an A-subbimodule of $\mathfrak{C}$.*

(i) *$\mathfrak{I}$ is a subbicomodule of $\mathfrak{C}$ if and only if $\mathfrak{I}$ is a (B, B')-subbimodule of $\mathfrak{C}$.*

(ii) *If $\mathfrak{I}$ is pure both as a left A-submodule and a right A-submodule of $\mathfrak{C}$, then $\mathfrak{I}$ is a subcoring of $\mathfrak{C}$ if and only if it is a (B, B')-subbimodule.*

For a left $\mathfrak{C}$-comodule M, define $C(M)$ as the sum of the images of all comodule homomorphisms from M to $\mathfrak{C}$. In presence of left and right rational pairings $T = (B, \mathfrak{C}, \langle -,-\rangle)$ and $T' = (\mathfrak{C}, B', [-,-])$, it is easy to prove that $C(M)$ is a subbicomodule of $\mathfrak{C}$. In fact, by definition, it is a left B-submodule of the (B, B')-bimodule $\mathfrak{C}$. Now if $b' \in B'$ and $c = f(m)$ for some homomorphism of left $\mathfrak{C}$-comodules, then $cb' = (r_{b'} \circ f)(m)$, where $r_{b'}$: $\mathfrak{C} \to \mathfrak{C}$ is the homomorphism of left comodules given by right multiplication by b'. Therefore, $C(M)$ is a (B, B')-subbimodule of $\mathfrak{C}$, and by Corollary 2.11, it is a subbicomodule of $\mathfrak{C}$, which will be called the *bicomodule of coefficients* of M.

Proposition 2.12. *Let $\lambda_M : M \to \mathfrak{C}\otimes_A M$ be a left comodule and assume there are rational pairings $T = (B, \mathfrak{C}, \langle -,-\rangle)$ and $T' = (\mathfrak{C}, B', [-,-])$ on the left and right, respectively.*

(i) *If $\tau : \mathfrak{I} \hookrightarrow \mathfrak{C}$ is a monomorphism of $\mathfrak{C}$-bicomodules such that $\lambda_M(M) \subseteq (\tau\otimes_A M)(\mathfrak{I}\otimes_A M)$, then $C(M) \subseteq \mathfrak{I}$.*

(ii) *If N is a subcomodule of M, then $C(N) \subseteq C(M)$ and $C(M/N) \subseteq C(M)$.*

(iii) *If $N \cong M$ is an isomorphism of comodules, then $C(N) = C(M)$.*

Proof. Let $c = f(m) \in C(M)$, where $f : M \to \mathfrak{C}$ is a homomorphism of left comodules, and write $\lambda_M(m) = \sum c_i \otimes m_i$ for some $c_i \in \mathfrak{J}$ and $m_i \in M$. Since f is a comodule map, we have

$$\Delta(c) = \Delta(f(m)) = (\mathfrak{C} \otimes_A f)(\lambda_M)(m) = \sum c_i \otimes f(m_i),$$

whence $c = \sum c_i \epsilon_{\mathfrak{C}}(f(m_i)) \in \mathfrak{J}$ by the counital property. This proves (i). Statements (ii) and (iii) are easy consequences of the definition of the bicomodule of coefficients. □

3 Semisimple Corings

We study the simplest kind of corings from the categorical point of view, namely, those corings having a semisimple category of comodules. We prove generalizations of known theorems for coalgebras and rings. In particular, we get a (unique) decomposition of any semisimple coring in terms of simple components. The structure of these simple components, which in the cases of rings and coalgebras over fields is described in terms of matrices, seems to be much more tangled in the present general setting (see, however, the last section for a structure theorem for the case of simple semisimple corings having a grouplike element).

Let $\mathfrak{C}$ be an A-coring. A right $\mathfrak{C}$-comodule M is said to be *simple* if M has no non-trivial factor comodules, and M is said to be *semisimple* if it is a direct sum of simple comodules.

Theorem 3.1. *Let $\mathfrak{C}$ be an A-coring. The following statements are equivalent:*

(i) *Every left $\mathfrak{C}$-comodule is semisimple and ${}^{\mathfrak{C}}\mathcal{M}$ is abelian.*
(ii) *Every right $\mathfrak{C}$-comodule is semisimple and $\mathcal{M}^{\mathfrak{C}}$ is abelian.*
(iii) *$\mathfrak{C}$ is semisimple as a left $\mathfrak{C}$-comodule and $\mathfrak{C}_A$ is flat.*
(iv) *$\mathfrak{C}$ is semisimple as a right $\mathfrak{C}$-comodule and ${}_A\mathfrak{C}$ is flat.*
(v) *$\mathfrak{C}$ is semisimple as a right $\mathfrak{C}^*$-module and $\mathfrak{C}_A$ is projective.*
(vi) *$\mathfrak{C}$ is semisimple as a left ${}^*\mathfrak{C}$-module and ${}_A\mathfrak{C}$ is projective.*

Proof. (i)⇒(iii). We need to prove that $\mathfrak{C}_A$ is flat. Accordingly with Proposition 1.2, it is enough to show that the underlying functor $U : {}^{\mathfrak{C}}\mathcal{M} \to {}_A\mathcal{M}$ is (left) exact. But this is the case since every monomorphism in ${}^{\mathfrak{C}}\mathcal{M}$ splits as all its objects are semisimple.

(iii)⇒(vi). By Proposition 1.2, ${}^{\mathfrak{C}}\mathcal{M}$ is abelian and the forgetful functor $U : {}^{\mathfrak{C}}\mathcal{M} \to {}_A\mathcal{M}$ is exact. Moreover, it has an exact right adjoint $\mathfrak{C} \otimes_A -$, which implies that it preserves projective objects. By Corollary 1.3, every left $\mathfrak{C}$-comodule is semisimple and thus projective in ${}^{\mathfrak{C}}\mathcal{M}$. Therefore, every left $\mathfrak{C}$-comodule is projective as a left A-module, and in particular, ${}_A\mathfrak{C}$ is projective. By Corollary 2.7, the category $\mathcal{M}^{\mathfrak{C}}$ of all right $\mathfrak{C}$-comodules is isomorphic to the category $\mathrm{Rat}({}_{^*\mathfrak{C}}\mathcal{M})$ of rational left ${}^*\mathfrak{C}$-modules. Moreover, since $\mathfrak{C}$ is a semisimple object in the Grothendieck category ${}^{\mathfrak{C}}\mathcal{M}$, it follows that $\mathfrak{C}$ is semisimple as a left module over its endomorphism ring. Now the map sending every endomorphism $f : \mathfrak{C} \to \mathfrak{C}$ to $\epsilon \circ f : \mathfrak{C} \to A$ gives a ring isomorphism $\mathrm{End}({}_{\mathfrak{C}}\mathfrak{C}) \cong {}^*\mathfrak{C}$. Therefore, ${}_{{}^*\mathfrak{C}}\mathfrak{C}$ is semisimple.

(vi)⇒(ii). By the version of Corollary 2.7 for right comodules, $\mathcal{M}^{\mathfrak{C}}$ is isomorphic to the category $\mathrm{Rat}^{T'}({}_{{}^*\mathfrak{C}}\mathcal{M})$ of all rational left ${}^*\mathfrak{C}$-comodules associated to the right rational pairing $T' = (\mathfrak{C}, {}^*\mathfrak{C}^{op}, [-,-])$ given in Example 2.5. Thus, $\mathcal{M}^{\mathfrak{C}}$ becomes an abelian category generated by the semisimple object ${}_{{}^*\mathfrak{C}}\mathfrak{C}$, whence every right $\mathfrak{C}$-comodule is semisimple.

(ii)⇒(iv). This is symmetric to (i)⇒(iii).

(iv)⇒(v). This is symmetric to (iii)⇒(vi).

(v)⇒(i). This is symmetric to (vi)⇒(ii). □

Definition 3.2. An A-coring satisfying the equivalent conditions in Theorem 3.1 will be called a *semisimple* coring.

Remark 3.3. Theorem 3.1 shows that if $\mathfrak{C}_A$ is flat and ${}_{\mathfrak{C}}\mathfrak{C}$ is semisimple, then ${}_A\mathfrak{C}$ is flat (even projective). This is not true for general corings; in fact, the coring I in Example 1.1 is finitely generated and projective as a right A-module but ${}_AI$ is not flat unless ${}_RB$ is. This is in connection with the following problem. If ${}^{\mathfrak{C}}\mathcal{M}$ is an abelian category, is then $\mathcal{M}^{\mathfrak{C}}$ abelian? Theorem 3.1 gives a positive answer, but we do not know if this is correct in general.

Remark 3.4. Theorem 3.1 can be applied to a wide rank of examples of categories of modules. For instance, let A be a ring graded by a group G. Then the category *gr-A* of all graded right A-modules is a category of Doi–Koppinen modules (see [10]), and it is in fact isomorphic to the category $\mathcal{M}^{\mathfrak{C}}$, where $\mathfrak{C} = A \otimes_{\mathbb{Z}} \mathbb{Z}[G]$ is the A-coring defined by the entwining structure $\psi : \mathbb{Z}[G] \otimes_{\mathbb{Z}} A \to A \otimes_{\mathbb{Z}} \mathbb{Z}[G]$ which maps $g \otimes a$ to $a \otimes g\deg(a)$ for any homogeneous element a. It turns out that ${}^{\mathfrak{C}}\mathcal{M}$ is isomorphic to the category *A-gr* of all G-graded left A-modules. Therefore, Theorem 3.1 says that *gr-A* is a semisimple category if and only if *A-gr* is a semisimple category.

The always marvelous Wedderburn–Artin structure theorem for semisimple artinian rings reposes upon a unique decomposition of the ring as a direct sum of simple artinian rings. This 'abstract' part of that classical result holds in the present setting. We first define the natural notions of simple corings and semiartinian corings.

Definition 3.5. A coring is said to be *simple* if it does not contain non-trivial subbicomodules. Every simple ring A is a simple A-coring with the trivial structure $A \cong A \otimes_A A$. Note that if $\mathfrak{C}$ is a semisimple coring, then it is simple if and only if it does not contain non-trivial subcorings.

Definition 3.6. Assume the category of all left comodules over an A-coring

$\mathfrak{C}$ is a Grothendieck category. The coring $\mathfrak{C}$ is said to be left *semiartinian* if it is semiartinian as an object in ${}^{\mathfrak{C}}\mathcal{M}$, namely, every factor comodule of ${}_{\mathfrak{C}}\mathfrak{C}$ contains a (non-zero) simple subcomodule.

The following characterization of simple semiartinian corings generalizes known facts concerning simple artinian rings.

Theorem 3.7. *Let $\mathfrak{C}$ be an A-coring. Assume the modules ${}_A\mathfrak{C}$ and $\mathfrak{C}_A$ are projective. The following statements are equivalent:*

(i) *$\mathfrak{C}$ is a simple left semiartinian coring.*
(ii) *$\mathfrak{C}$ is a simple coring and contains a simple left $\mathfrak{C}$-subcomodule.*
(iii) *$\mathfrak{C}$ is a semisimple coring with a unique type of simple left $\mathfrak{C}$-comodule.*
(iv) *$\mathfrak{C}$ is a simple right semiartinian coring.*
(v) *$\mathfrak{C}$ is a simple coring and contains a simple right $\mathfrak{C}$-subcomodule.*
(vi) *$\mathfrak{C}$ is a semisimple coring with a unique type of simple right $\mathfrak{C}$-comodule.*

Proof. (i)$\Rightarrow$(ii). It is obvious.

(ii)$\Rightarrow$(iii). Let S be a simple left subcomodule of $\mathfrak{C}$. By Corollary 2.7, S is a simple right $\mathfrak{C}^*$-submodule of $\mathfrak{C}$. Now ${}^*\mathfrak{C}S$ is a non-zero $({}^*\mathfrak{C}, \mathfrak{C}^*)$-subbimodule of $\mathfrak{C}$ which, by Corollary 2.11, is a non-zero subbicomodule. Hence, ${}^*\mathfrak{C}S = \mathfrak{C}$, and therefore, $\mathfrak{C}$ is a sum of homomorphic images of the simple right $\mathfrak{C}^*$-module S. Apply Corollary 2.7.

(iii)$\Rightarrow$(i). Obviously, every semisimple coring is left semiartinian. Let $\mathfrak{I}$ be a non-zero subbicomodule of $\mathfrak{C}$. In particular, $\mathfrak{I}$ is a left $\mathfrak{C}$-subcomodule of $\mathfrak{C}$ so that it contains a simple subcomodule S. By Proposition 2.12(i)(ii), $C(S) \subseteq C(\mathfrak{I}) = \mathfrak{I}$. Since $\mathfrak{C}$ is isomorphic to a direct sum of copies of S, we apply Proposition 2.12(iii) to obtain $C(S) = \mathfrak{C}$. Hence, $\mathfrak{I} = \mathfrak{C}$ and $\mathfrak{C}$ is simple.

(ii)$\Rightarrow$(v). We have already proved that if $\mathfrak{C}$ is simple and contains a simple left comodule, then $\mathfrak{C}$ is semisimple. Thus, it contains a simple right $\mathfrak{C}$-comodule.

Finally, (iv), (v), (vi) are proved to be equivalent in an analogous way to the proof of the equivalence between (i), (ii), (iii); which also allows to derive that (v) implies (ii). This finishes the proof. □

Remark 3.8. Let $\mathfrak{C}$ be a simple semiartinian A-coring. By Theorem 3.7, we have ${}_{\mathfrak{C}}\mathfrak{C} \cong S^{(\Xi)}$, where S is a simple left $\mathfrak{C}$-comodule and Ξ is an index set. In contrast with the coalgebra or ring cases (i.e., when $\mathfrak{C}$ is a coalgebra over a field or $\mathfrak{C} = A$), the set Ξ is not necessarily finite. In fact, consider the A-coring I given in Example 1.1 with R a simple artinian ring, B the coproduct of Ξ copies of the unique simple left R-module (Ξ is any infinite set) and S is the endomorphism ring of the left R-module B. Then I is a simple semiartinian ring and it is isomorphic to a direct sum of infinitely many copies of a simple left I-comodule (which is essentially the unique simple left R-module). This example also shows that the basis ring A is not necessarily semisimple or even artinian for a semisimple A-coring.

We finish this section by showing that semisimple corings can be completely described in terms of simple semiartinian (or simple semisimple) corings.

Theorem 3.9. *An A-coring* $\mathfrak{C}$ *is semisimple if and only if it decomposes as* $\mathfrak{C} = \oplus_{\alpha\in\Lambda}\mathfrak{C}_\alpha$, *where* $\mathfrak{C}_\alpha$ *is a simple semisimple A-subcoring for every* $\alpha \in \Lambda$. *In such a case, the decomposition is unique.*

Proof. Assume $\mathfrak{C}$ is semisimple. Let Λ be a set of representatives of all simple right $\mathfrak{C}$-comodules. For each $\alpha \in \Lambda$, define $\mathfrak{C}_\alpha$ to be the αth isotypic component of $\mathfrak{C}_\mathfrak{C}$. Since $\mathfrak{C}$ is right semisimple, it follows that $\mathfrak{C} = \oplus_{\alpha\in\Lambda}\mathfrak{C}_\alpha$. We know from Corollary 2.7 that $\mathfrak{C}_\alpha$ is a left ${}^*\mathfrak{C}$-submodule of $\mathfrak{C}$. Given $c^* \in \mathfrak{C}^*$, its right multiplication map is a homomorphism of left ${}^*\mathfrak{C}$-modules, and thus, of right $\mathfrak{C}$-comodules. It follows that $\mathfrak{C}_\alpha$ is a right $\mathfrak{C}^*$-submodule of $\mathfrak{C}$, and by Corollary 2.11, $\mathfrak{C}_\alpha$ is a subcoring of $\mathfrak{C}$. Obviously, $\mathfrak{C}_\alpha$ is semisimple with a unique type of simple; by Theorem 3.7, $\mathfrak{C}_\alpha$ is a simple semiartinian A-coring. Finally, the converse implication is easily deduced from the fact that given the stated decomposition $\mathfrak{C} = \oplus_{\alpha\in\Lambda}\mathfrak{C}_\alpha$, the right $\mathfrak{C}$-subcomodules of $\mathfrak{C}$ are of the form $\oplus_{\alpha\in\Lambda}M_\alpha$, where M_α is a $\mathfrak{C}_\alpha$-subcomodule of $\mathfrak{C}_\alpha$ for every α. The uniqueness comes from the observation that these $\mathfrak{C}_\alpha$ are just the isotypic components of $\mathfrak{C}_\mathfrak{C}$. □

4 Simple Semiartinian Corings with a Grouplike Element

A complete description of all semisimple corings over a given ring A would be obtained, in view of Theorem 3.9, throughout the knowledge of the structure of simple semiartinian A-corings. The structure of a general simple semiartinian coring seems to be quite intricate (see Remark 3.8). It is possible, however, to recognize the simple semiartinian A-corings having a grouplike element as the canonical corings $A \otimes_B A$, where B runs the set of simple artinian subrings of A, as we will prove in this section.

Let $\mathfrak{C}$ be an A-coring. A non-zero element $g \in \mathfrak{C}$ such that $\epsilon(g) = 1$ and $\Delta(g) = g \otimes_A g$ is called a *grouplike* element. An example of corings with such an element is $A \otimes_B A$ cited in Example 1.4 taking $g = 1 \otimes_B 1$.

Lemma 4.1. [4, Lemma 5.1] *Let* $\mathfrak{C}$ *be an A-coring. Then A is a right* $\mathfrak{C}$*-comodule if and only if A is a left* $\mathfrak{C}$*-comodule, if and only if there exists a grouplike element* $g \in \mathfrak{C}$. *In that case, the left and right coactions are given by* $\lambda_A : A \to \mathfrak{C}$, $a \mapsto ag \otimes_A 1$ *and* $\rho_A : A \to \mathfrak{C}$, $a \mapsto 1 \otimes_A ga$.

Assume $\mathfrak{C}$ has a grouplike element g, and consider the *subring of coinvariants* of A defined by $A^g = \{a \in A \mid ag = ga\}$; this ring is isomorphic to $\mathrm{End}(A_\mathfrak{C})$, and also to $\mathrm{End}({}_\mathfrak{C}A)$. Then we have a functor [4, Proposition 5.2] $(-)^g : \mathcal{M}^\mathfrak{C} \to \mathcal{M}_{A^g}$, which assigns to every right $\mathfrak{C}$-comodule M the right A^g-module of *coinvariants*

$$M^g = \{m \in M \mid \rho_M(m) = m \otimes_A g\}.$$

It is easily shown that this functor is naturally isomorphic to the functor $\mathrm{Hom}_{\mathfrak{C}}(A_{\mathfrak{C}}, -)$. The analogous discussion is pertinent for left $\mathfrak{C}$-comodules.

Proposition 4.2. *Let $B \to A$ be a ring extension, and $A \otimes_B A$ with the canonical A-coring structure be defined in Example 1.4. Assume B is a simple artinian ring. Then $A \otimes_B A$ is a simple semisimple A-coring and $A^{1 \otimes_B 1} = B$. Moreover, A is a simple left (or right) $A \otimes_B A$-comodule if and only if B is a division ring.*

Proof. We know that A_B and $_BA$ are projective modules, and this implies that the coring $\mathfrak{C} = A \otimes_B A$ is projective as a left and as a right A-module. By Corollary 2.7, the category $^{A \otimes_B A}\mathcal{M}$ is isomorphic to the category of all rational right $\mathfrak{C}^*$-modules. Recall from [14, Example 3.3] that there is an anti-isomorphism of rings

$$\begin{array}{ccc} \mathfrak{C}^* & \longrightarrow & \mathrm{End}(A_B) \\ g \longmapsto [a \mapsto g(a \otimes_B 1)], & & \epsilon \circ (f \otimes_B A) \longleftarrow\!\dashv f. \end{array}$$

Some straightforward computations show that the canonical left $\mathrm{End}(A_B)$-module structure of A corresponds to the structure of (rational) right $\mathfrak{C}^*$-module, whenever the coactions of A are derived from $1 \otimes_B 1$ (see Lemma 4.1). Since A_B is a homogeneous semisimple right B-module, it follows that $_{\mathrm{End}(A_B)}A$ is homogeneous semisimple too. We conclude by Lemma 4.1 and Theorem 2.6 that A is a direct sum of copies of a simple left $\mathfrak{C}$-comodule. Now let $a \in A$ and consider the homomorphism $\phi_a : A \to \mathfrak{C}$ of left A-modules given by $a' \mapsto a' \otimes_B a$, which is, in fact, a homomorphism of left $\mathfrak{C}$-comodules. It follows that A generates $A \otimes_B A$ as a left comodule. In particular, $A \otimes_B A$ is a sum of copies of a simple comodule which, in the light of Theorem 3.7, shows that $A \otimes_B A$ is a simple semisimple A-coring. Finally,

$$A^{1 \otimes_B 1} \cong \mathrm{End}(_{A \otimes_B A}A) = \mathrm{End}(A_{\mathfrak{C}^*}) = \mathrm{End}(_{\mathrm{End}(A_B)}A) = \mathrm{Biend}(A_B), \quad (5)$$

and A_B is a balanced B-module (remember that B is simple artinian). Hence, $B = A^{1 \otimes_B 1}$. The direct implication in the stated equivalence is obvious, in view of (5). Conversely, suppose B is a division ring. Then $A \otimes_B A$ is a simple semisimple A-coring; in particular, each monomorphism of left $A \otimes_B A$-comodules splits. Since $B \cong \mathrm{End}(_{A \otimes_B A}A)$ by (5), $_{A \otimes_B A}A$ is a simple comodule. □

The next theorem tells us that Proposition 4.2 gives all possible examples of simple semisimple corings with a grouplike element.

Theorem 4.3. *Let $\mathfrak{C}$ be an A-coring and $g \in \mathfrak{C}$ a grouplike element. Assume $\mathfrak{C}$ is a simple semisimple A-coring. Then A^g is a simple artinian ring and the canonical A-bimodule map $A \otimes_{A^g} A \to \mathfrak{C}$ sending $1 \otimes_{A^g} 1$ to*

g is an isomorphism of A-corings. Moreover, A is a simple left (or right) $\mathfrak{C}$-comodule if and only if A^g is a division ring.

Proof. Endow A with the structure of a right $\mathfrak{C}$-comodule derived from g. Since $\mathfrak{C}$ is assumed to be simple semisimple, A is a direct sum of copies of the only simple right $\mathfrak{C}$-comodule. Moreover, this direct sum, being of right A-submodules after all, is finite. Therefore, $\mathrm{End}(A_{\mathfrak{C}}) \cong A^g$ is a simple artinian ring. It is easily proved that the A-bimodule homomorphism $\varphi : A \otimes_{A^g} A \to \mathfrak{C}$ sending $a \otimes a'$ to aga' is an A-coring homomorphism. Since $A_{\mathfrak{C}}$ is a finitely generated projective generator for $\mathcal{M}^{\mathfrak{C}}$, a standard consequence of Gabriel–Popescu's Theorem (see, e.g., [11, Corollary 9.7]) says that $\mathrm{Hom}_{\mathfrak{C}}(A, -) : \mathcal{M}^{\mathfrak{C}} \to \mathcal{M}_{\mathrm{End}(A_{\mathfrak{C}})}$ is an equivalence of categories. Now $\mathrm{Hom}_{\mathfrak{C}}(A, -) \cong (-)^g$ naturally, which implies by [4, Theorem 5.6] that φ is an isomorphism. The last statement is proved as in the proof of Proposition 4.2. □

Following [4, Definition 5.3], we say that an A-coring with grouplike g is *Galois* if the A-coring map sending $1 \otimes_{A^g} 1$ to g gives an isomorphism $\mathfrak{C} \cong A \otimes_{A^g} A$. Thus, Theorem 4.3 says that every simple semisimple A-coring with a grouplike element is Galois. We have already more, as the following theorem shows, which collects the relevant information about simple semisimple corings with a grouplike element.

Theorem 4.4. *The following conditions are equivalent for an A-coring $\mathfrak{C}$ with a grouplike element g:*

(i) *$\mathfrak{C}$ is a simple semisimple A-coring.*
(ii) *$\mathfrak{C} \cong A \otimes_B A$ for some simple artinian subring B of A.*
(iii) *$\mathfrak{C}$ is Galois and A^g is a simple artinian ring.*
(iv) *$\mathfrak{C}_A$ is flat, A is a projective generator in ${}^{\mathfrak{C}}\mathcal{M}$, and A^g is a simple artinian ring.*
(v) *${}_A\mathfrak{C}$ is flat, A is a projective generator in $\mathcal{M}^{\mathfrak{C}}$, and A^g is a simple artinian ring.*

Proof. (i)⇒(iii). This is Theorem 4.3.

(iii)⇒(ii). Obvious.

(ii)⇒(i). It follows from Proposition 4.2.

(i)⇒(iv). By Theorem 3.1, $\mathfrak{C}_A$ is in fact projective. Theorem 3.7 gives that every left $\mathfrak{C}$-comodule is a direct sum of copies of the unique simple comodule. Thus, every non-zero comodule is a projective generator for ${}^{\mathfrak{C}}\mathcal{M}$. Finally, since (i) is equivalent to (iii), we know that A^g is a simple artinian ring.

(iv)⇒(iii). The proof of Theorem 4.3 is easily adapted to obtain this implication.

(v)⇔(iv). It follows by symmetry. □

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